

WEIGHTED SUB-LAPLACIANS ON MÉTIVIER GROUPS: ESSENTIAL SELF-ADJOINTNESS AND SPECTRUM

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ABSTRACT

Let G be a Métivier group and let N be any homogeneous norm on G . For $\alpha > 0$ denote by w_α the function e^{-N^α} and consider the weighted sub-Laplacian \mathcal{L}^{w_α} associated with the Dirichlet form $\phi \mapsto \int_G |\nabla_{\mathcal{H}} \phi(y)|^2 w_\alpha(y) dy$, where $\nabla_{\mathcal{H}}$ is the horizontal gradient on G . Consider \mathcal{L}^{w_α} with domain C_c^∞ . We prove that \mathcal{L}^{w_α} is essentially self-adjoint when $\alpha \geq 1$. For a particular N , which is the norm appearing in \mathcal{L} 's fundamental solution when G is an H-type group, we prove that \mathcal{L}^{w_α} has purely discrete spectrum if and only if $\alpha > 2$, thus proving a conjecture of J. Inglis.

1. INTRODUCTION AND PRELIMINARY RESULTS

Let G be a stratified group. Denote by dy its left- and right-invariant Haar measure and by $\nabla_{\mathcal{H}}$ the horizontal gradient on G . Given a positive measurable function w on G , denote by $L^2(G, w) =: L^2(w)$ the space of all functions on G that are square integrable with respect to the measure $w(y) dy$, with consequent inner product. We define the weighted sub-Laplacian \mathcal{L}^w as the operator associated with the Dirichlet form

$$\phi \mapsto \int_G |\nabla_{\mathcal{H}} \phi(y)|^2 w(y) dy$$

for every $\phi \in C_c^\infty$. Under very weak assumptions on w , such form is symmetric, densely defined, accretive and continuous; thus the operator \mathcal{L}^w , with domain C_c^∞ , has self-adjoint extensions. We consider in particular the family of weights

$$(w_\alpha = e^{-N^\alpha})_{\alpha > 0}$$

where N is any homogeneous norm on G . When G is some \mathbb{R}^n , $\alpha = 2$ and the homogeneous norm is the Euclidean norm on \mathbb{R}^n , this operator is indeed – up to normalization constants – the classical Ornstein-Uhlenbeck operator; therefore \mathcal{L}^{w_2} , for some choice on N , can be considered as one of its possible sub-Riemannian versions on stratified groups. \mathcal{L}^{w_α} for $\alpha > 0$ is then its natural generalization.

The aim of this paper is to investigate the essential self-adjointness of \mathcal{L}^{w_α} and, in the particular case when G is a Métivier group, whether its self-adjoint extensions have purely discrete spectra. We recall that a self-adjoint operator T on a Hilbert space has purely discrete spectrum if its spectrum $\sigma(T)$ is discrete, and for each $\lambda \in \sigma(T)$ the range of the spectral projection $E(\{\lambda\})$ is finite-dimensional (cf. [14]). When G is an H-type group (a subclass of Métivier groups) and N is either the Carnot-Charathéodory norm or the Kaplan norm, the discreteness of the spectrum of \mathcal{L}^{w_α} has been investigated by J. Inglis [7]. By using super Poincaré inequalities he proved that when N is the Carnot-Charathéodory norm \mathcal{L}^{w_α} has discrete spectrum for every $\alpha > 1$. When N is the Kaplan norm he proved that the spectrum of \mathcal{L}^{w_α} is non-discrete for $1 < \alpha < 2$. Even though his technique, based on functional inequalities, did not allow him to investigate the spectrum of \mathcal{L}^{w_α} for the Kaplan norm when $\alpha \geq 2$, Inglis conjectured that the spectrum is discrete when $\alpha > 2$. By using different techniques, we completely

characterize the discreteness of the spectrum of \mathcal{L}^{w_α} for every $\alpha > 0$ in the case of the Kaplan norm, not only solving Inglis's conjecture, but also giving a direct proof of the non-discreteness of the spectrum when $0 < \alpha \leq 2$, thus improving Inglis's result.

The content of this paper is as follows. In the remaining of this section we fix the notation and recall the basic features of analysis on stratified groups. In Section 2 we prove a general criterion (Theorem 2.3) for establishing existence and uniqueness of self-adjoint extensions of \mathcal{L}^w , with domain C_c^∞ , on $L^2(w)$. It is based on suitable integrability or (essential) boundedness properties of the weight w and its horizontal derivatives. As a corollary, the essential self-adjointness of \mathcal{L}^{w_α} when $\alpha \geq 1$ (Theorem 2.7) is obtained. By using a well known technique (see e.g. [5, 3, 7]), in Section 3 we then show how to reduce the study of \mathcal{L}^w to the study of the Schrödinger operator $\mathcal{L} + V$ as an operator on $L^2(G, dy)$, where the potential V is

$$V = -\frac{1}{4} \frac{|\nabla_{\mathcal{H}} w|^2}{w^2} - \frac{1}{2} \frac{\mathcal{L} w}{w}.$$

In Section 4, we restrict our analysis to Métivier groups and the case when N is the Kaplan norm. We prove that the spectrum of \mathcal{L}^{w_α} is purely discrete if and only if $\alpha > 2$, thus proving Inglis's conjecture [7]. To prove the theorem we reduce matters to studying the spectrum of the Schrödinger operator $\mathcal{L} + V_\alpha$ obtained by conjugating \mathcal{L}^{w_α} with a suitable isometry. We then apply a generalization to stratified groups of a theorem of B. Simon relating the discreteness of the spectrum to the behaviour of the sublevel sets of the potential V_α [17].

1.1. Preliminaries. A stratified group G is a connected, simply connected Lie group whose Lie algebra \mathfrak{g} admits a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\nu \tag{1.1}$$

such that $[\mathfrak{g}_1, \mathfrak{g}_{k-1}] = \mathfrak{g}_k$ for every $k \leq \nu$, and $[\mathfrak{g}_1, \mathfrak{g}_\nu] = 0$. The first layer \mathfrak{g}_1 is often referred to as *horizontal layer*. We set $n_k = \dim \mathfrak{g}_k$; $Q = \sum_{k=1}^\nu k n_k$ will denote the homogeneous dimension of G . We shall write G^* instead of $G \setminus \{e\}$.

The Lie algebra \mathfrak{g} is canonically endowed with the family of dilations $\delta_r(\sum_{k=1}^\nu Y_k) = \sum_{k=1}^\nu r^k Y_k$, where $Y_k \in \mathfrak{g}_k$. One can then define a family of dilations on G via the exponential map, by $\delta_r(\exp Y) = \exp \delta_r(Y)$.

Recall that G is unimodular, so that it can be endowed with a Haar measure dy which is both left and right invariant. Thus, one can define the space L^p as the space of (equivalence classes of) measurable functions f such that $|f|^p$ is integrable with respect to dy , with the usual norm. The space $L^p(\mathfrak{g}_1)$ of p -th power integrable \mathfrak{g}_1 -valued functions is defined analogously.

From now on, we fix a basis $(X_j)_{1 \leq j \leq n_1}$ of left-invariant vector fields for \mathfrak{g}_1 . We endow \mathfrak{g}_1 with the unique inner product which turns (X_j) into an orthonormal basis; if f is a differentiable function on G , its horizontal gradient is the vector field

$$\nabla_{\mathcal{H}} f := \sum_{j=1}^{n_1} (X_j f) X_j.$$

Observe that, if f, g are smooth functions on G , then $(\nabla_{\mathcal{H}} f)(g) = \nabla_{\mathcal{H}} f \cdot \nabla_{\mathcal{H}} g$, where the right hand side is the inner product of the vector fields $\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g$.

For a vector-valued differentiable function $F = (F_1, \dots, F_{n_1})$, its horizontal divergence is the function

$$\operatorname{div}_{\mathcal{H}} F = \sum_{j=1}^{n_1} X_j F_j,$$

and the sub-Laplacian \mathcal{L} will then be the (positive) operator

$$\mathcal{L} := -\operatorname{div}_{\mathcal{H}} \nabla_{\mathcal{H}} = -\sum_{j=1}^{n_1} X_j^2.$$

With domain C_c^∞ , \mathcal{L} is well known to be essentially self-adjoint on L^2 . With a slight abuse of notation, we shall denote by \mathcal{L} also its closure and the operator interpreted in the sense of distributions. $\operatorname{Dom}(\mathcal{L})$ will stand for the set of $f \in L^2$ such that $\mathcal{L}f$ also belongs to L^2 .

Definition 1.1 (cf. [6]). For every $k \in \mathbb{Z}$ define S_k^2 as the completion of C_c^∞ with respect to the norm

$$f \mapsto \|(I + \mathcal{L})^{k/2} f\|_{L^2},$$

and

$$S_{k,\operatorname{loc}}^2 = \{f \in \mathcal{D}' : \phi f \in S_k^2 \ \forall \phi \in C_c^\infty\}.$$

In the following lemma we state some general properties of the Sobolev spaces just introduced, which will be used later on. They are the analogues for $S_{k,\operatorname{loc}}^2$ of some properties of S_k^2 that can be found in [6]. The proof is omitted.

Lemma 1.2. *The following hold:*

- (i) $S_{1,\operatorname{loc}}^2 = \{f \in L_{\operatorname{loc}}^2 : \nabla_{\mathcal{H}} f \in L_{\operatorname{loc}}^2\}$ and $S_{2,\operatorname{loc}}^2 = \{f \in S_{1,\operatorname{loc}}^2 : \mathcal{L}f \in L_{\operatorname{loc}}^2\}$;
- (ii) If $f \in L_{\operatorname{loc}}^2$, then $Xf \in S_{-1,\operatorname{loc}}^2$ for every $X \in \mathfrak{g}_1$;
- (iii) If $f \in L_{\operatorname{loc}}^2$ is such that $\mathcal{L}f \in S_{-1,\operatorname{loc}}^2$, then $f \in S_{1,\operatorname{loc}}^2$.

We shall also need the following proposition. It is the analogue for the sub-Laplacian on G of a well known inequality, due to Kato [10], involving the Laplacian on \mathbb{R}^n . We recall that, if T_1, T_2 are two distributions, $T_1 \leq T_2$ means that $T_2 - T_1$ is a positive (Radon) measure. The sign function on \mathbb{R}^d , $d \geq 1$, or \mathbb{C} is defined as $\frac{z}{|z|}$ when $z \neq 0$, and 0 when $z = 0$.

Proposition 1.3. *Let $f \in L_{\operatorname{loc}}^1$ such that $\mathcal{L}f \in L_{\operatorname{loc}}^1$. Then $\mathcal{L}|f| \leq \operatorname{Re} \left[\overline{\operatorname{sign}(f)} \mathcal{L}f \right]$.*

Its proof is a straightforward adaptation to our setting of [15, Theorem X.27], and is omitted. A similar statement, in a much more general context but with stronger hypotheses, can be found in [16].

2. ESSENTIAL SELF-ADJOINTNESS OF WEIGHTED SUB-LAPLACIANS

In this section, G will be a stratified *non commutative* group, so that \mathfrak{g} has at least two layers and $Q \geq 4$. Henceforth, we adopt the well known notation of writing (\cdot, \cdot) for the inner product of elements in some Hilbert space (L^2 or $L^2(w)$, see Definition 2.2), while $\langle \cdot, \cdot \rangle$ will stand for (sesquilinear) distributional pairings.

Definition 2.1. We shall call *admissible weight* a strictly positive and weakly differentiable function w on G , such that $w, w^{-1} \in L_{\operatorname{loc}}^\infty$, and $\nabla_{\mathcal{H}} w \in L_{\operatorname{loc}}^2$.

Definition 2.2. Let w be an admissible weight. We denote by $L^2(w)$ the Hilbert space

$$\{f : G \rightarrow \mathbb{C} : f \text{ measurable, } \int_G |f(y)|^2 w(y) dy < \infty\}$$

with the usual scalar product and norm. The space $L^2(w; \mathfrak{g}_1)$ of square-integrable \mathfrak{g}_1 -valued functions is defined analogously. Moreover, we denote by

$$S_1^2(w) = \{f \in L^2(w) : \nabla_{\mathcal{H}} f \in L^2(w; \mathfrak{g}_1)\}$$

the weighted Sobolev space, endowed with the standard norm.

By routine arguments, one can prove that $S_1^2(w)$ is a Hilbert space and that C_c^∞ is dense in it. Therefore, the sesquilinear form

$$\mathfrak{t}(f, g) := (\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g)_{L^2(w)}, \quad \text{Dom}(\mathfrak{t}) = S_1^2(w)$$

is symmetric, densely defined, accretive, continuous and closed. By the general theory of sesquilinear forms (see, e.g., [9, IV, Theorem 2.6]) there exists a positive self-adjoint operator \mathcal{L}^w such that

$$\begin{aligned} \text{Dom}(\mathcal{L}^w) &= \{f \in S_1^2(w) : \exists \tilde{f} \in L^2(w) \forall g \in S_1^2(w) \quad (\tilde{f}, g)_{L^2(w)} = \mathfrak{t}(f, g)\}, \\ \mathcal{L}^w f &:= \tilde{f} \quad \forall f \in \text{Dom}(\mathcal{L}^w). \end{aligned}$$

An easy computation shows that for every $\phi, \psi \in C_c^\infty$

$$(\mathcal{L}^w \phi, \psi)_{L^2(w)} = \mathfrak{t}(\phi, \psi) = \left(\mathcal{L} \phi - \frac{\nabla_{\mathcal{H}} w}{w} \cdot \nabla_{\mathcal{H}} \phi, \psi \right)_{L^2(w)}.$$

It is then natural to define the operator

$$\mathcal{L}_0^w := \mathcal{L} - \frac{\nabla_{\mathcal{H}} w}{w} \cdot \nabla_{\mathcal{H}}, \quad \text{Dom}(\mathcal{L}_0^w) := C_c^\infty. \quad (2.1)$$

Notice that, since $\nabla_{\mathcal{H}} w \in L_{\text{loc}}^2$ and $w^{-1} \in L_{\text{loc}}^\infty$ by Definition 2.1, the operator \mathcal{L}_0^w maps C_c^∞ to $L^2(w)$. Since for every $f \in S_{1,\text{loc}}^2$, we have $\frac{\nabla_{\mathcal{H}} w}{w} \cdot \nabla_{\mathcal{H}} f \in L_{\text{loc}}^1$, we may define also the operator

$$\mathcal{L}_1^w := \mathcal{L} - \frac{\nabla_{\mathcal{H}} w}{w} \cdot \nabla_{\mathcal{H}} : S_{1,\text{loc}}^2 \rightarrow \mathcal{D}',$$

where the derivatives are to be understood in the distributional sense. The following theorem, whose proof will occupy the remainder of this section, describes the relations among the operators \mathcal{L}_0^w , \mathcal{L}^w and \mathcal{L}_1^w .

Theorem 2.3. *If w is an admissible weight, then \mathcal{L}^w is a positive self-adjoint extension of \mathcal{L}_0^w and $\text{Dom}(\mathcal{L}^w) = \{f \in S_1^2(w) : \mathcal{L}_1^w f \in L^2(w)\}$.*

Assume, in addition, that one of the following conditions holds:

- (a) $\nabla_{\mathcal{H}} w \in L_{\text{loc}}^\infty$ and $\mathcal{L} w \in L_{\text{loc}}^Q$;
- (b) $\frac{\nabla_{\mathcal{H}} w}{(1+N)w} \in L^\infty$ for some (hence any) homogeneous norm N on G .

Then \mathcal{L}_0^w is essentially self-adjoint, i.e. \mathcal{L}^w is the unique self-adjoint extension of \mathcal{L}_0^w .

Remark 2.4. The essential self-adjointness of \mathcal{L}_0^w is well known when the weight w is smooth (see e.g. [1, Proposition 3.2.1]). The classical argument makes use of two ingredients:

- (i) the hypoellipticity of \mathcal{L}_0^w : any $f \in \text{Dom}((\mathcal{L}_0^w)^*)$ such that $(\mathcal{L}_0^w)^* f + f = 0$ is smooth;
- (ii) completeness: there exists an increasing sequence (ψ_k) of positive functions in C_c^∞ that converges to 1 almost everywhere and $|\nabla_{\mathcal{H}} \psi_k| \leq 1/k$ for all $k \geq 1$.
We shall refer to the sequence (ψ_k) as an *approximate unity* for G .

Completeness does not depend on the weight and therefore holds also under our assumptions. It suffices to define $\psi_k = \psi \circ \delta_{1/k}$, $k \geq 1$, where ψ is a function in C_c^∞ such that $0 \leq \psi \leq 1$ and $\psi(x) = 1$ if $N(x) \leq 1$ and $\psi(x) = 0$ if $N(x) \geq 2$.

Instead, hypoellipticity of \mathcal{L}_0^w fails when the weight w is nonsmooth. However, we shall see that a weaker form of hypoellipticity holds if $\nabla_{\mathcal{H}} w \in L_{\text{loc}}^\infty$ and $\mathcal{L} w \in L_{\text{loc}}^Q$ (see Lemma 2.6).

To prove the theorem we shall need two lemmas. In their proofs and henceforth, we shall apply the common rules of derivation to products and compositions of distributions with Lipschitz functions. Each equality may be proved either by density arguments, or by convolution on the left with a suitable approximate identity.

Lemma 2.5. *Let w be an admissible weight. If*

- (i) *either $f, g \in S_1^2(w)$ and $\mathcal{L}_1^w f \in L^2(w)$*
- (ii) *or $f, g \in S_{1,\text{loc}}^2$, $\mathcal{L}_1^w f \in L_{\text{loc}}^2$, g has compact support and $\nabla_{\mathcal{H}} w \in L_{\text{loc}}^\infty$,*

then

$$(\mathcal{L}_1^w f, g)_{L^2(w)} = \langle \nabla_{\mathcal{H}} f, w \nabla_{\mathcal{H}} g \rangle.$$

Proof. We first assume (i). Since both sides of the asserted equality are continuous functions of $g \in S_1^2(w)$, by density it will be enough to prove our assertion when $g \in C_c^\infty$. If $\mathcal{L}_1^w f \in L^2(w)$, since $\nabla_{\mathcal{H}} f$ and $\frac{\nabla_{\mathcal{H}} w}{w} \in L_{\text{loc}}^2$, we infer that $\mathcal{L} f$ and $\frac{\nabla_{\mathcal{H}} w}{w} \cdot \nabla_{\mathcal{H}} f \in L_{\text{loc}}^1$. Therefore, $\mathcal{L}_1^w f = \mathcal{L} f - \frac{\nabla_{\mathcal{H}} w}{w} \cdot \nabla_{\mathcal{H}} f \in L_{\text{loc}}^1$ and the equality to prove is implied by:

$$\langle \mathcal{L} f, wg \rangle = \langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}}(wg) \rangle,$$

where the first pairing refers to the duality between L_{loc}^1 and the space of L^∞ functions with compact support, while the second pairing refers to the duality between L_{loc}^2 and the space of L^2 functions with compact support. Since $wg \in S_1^2$ and has compact support, an integration by parts shows that this identity holds for $f \in C^\infty$. The general case can be reduced to this, by convolving f on the left with an approximate identity (η_k) of smooth functions with compact support. Since \mathcal{L} and $\nabla_{\mathcal{H}}$ commute with left convolution, $\mathcal{L}(\eta_k * f) = \eta_k * \mathcal{L} f$ converges to $\mathcal{L} f$ in L_{loc}^1 and $\nabla_{\mathcal{H}}(\eta_k * f) = \eta_k * \nabla_{\mathcal{H}} f$ converges to $\nabla_{\mathcal{H}} f$ in L_{loc}^2 . The conclusion follows by the continuity of the pairings. This proves case (i).

Case (ii) may be reduced to case (i), by multiplying f by a function $\phi \in C_c^\infty$ such that $\phi = 1$ in a neighbourhood of the support of g and observing that $f\phi, g \in S_1^2(w)$, $\mathcal{L}_1^w(f\phi) = \phi \mathcal{L}_1^w f + f \mathcal{L}_1^w \phi - 2\nabla_{\mathcal{H}} \phi \cdot \nabla_{\mathcal{H}} f \in L^2(w)$, since $\nabla_{\mathcal{H}} w \in L_{\text{loc}}^\infty$, and

$$(\mathcal{L}_1^w f, g)_{L^2(w)} = (\mathcal{L}_1^w(f\phi), g)_{L^2(w)} = (\nabla_{\mathcal{H}}(f\phi), \nabla_{\mathcal{H}} g)_{L^2(w)} = (\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g)_{L^2(w)}.$$

□

Lemma 2.6. *Let w be an admissible weight and suppose that $f \in \text{Dom}((\mathcal{L}_0^w)^*)$ satisfies $(\mathcal{L}_0^w)^* f + f = 0$. Then*

- (i) *if $\nabla_{\mathcal{H}} w \in L_{\text{loc}}^\infty$ then $f \in S_{1,\text{loc}}^2$;*
- (ii) *if, furthermore, $\mathcal{L} w \in L_{\text{loc}}^Q$ then $f \in S_{2,\text{loc}}^2$.*

Proof. Since $\text{Dom}((\mathcal{L}_0^w)^*) \subseteq L^2(w) \subseteq L_{\text{loc}}^2$, we have that $f \in L_{\text{loc}}^2$; hence, both fw and $f\nabla_{\mathcal{H}} w$ belong to L_{loc}^2 . Then, for every $\phi \in C_c^\infty$,

$$\begin{aligned} \langle w(\mathcal{L}_0^w)^* f, \phi \rangle &= ((\mathcal{L}_0^w)^* f, \phi)_{L^2(w)} \\ &= (f, \mathcal{L}_0^w \phi)_{L^2(w)} \\ &= \left(f, \mathcal{L} \phi - \frac{\nabla_{\mathcal{H}} w}{w} \cdot \nabla_{\mathcal{H}} \phi \right)_{L^2(w)} \\ &= \langle \mathcal{L}(fw), \phi \rangle + \langle \text{div}_{\mathcal{H}}(f \nabla_{\mathcal{H}} w), \phi \rangle. \end{aligned}$$

By the arbitrariness of ϕ , this implies that

$$w(\mathcal{L}_0^w)^* f = \mathcal{L}(fw) + \text{div}_{\mathcal{H}}(f \nabla_{\mathcal{H}} w).$$

By Lemma 1.2, (ii), $\operatorname{div}_{\mathcal{H}}(f\nabla_{\mathcal{H}}w) \in S_{-1,\operatorname{loc}}^2$ since $f\nabla_{\mathcal{H}}w \in L_{\operatorname{loc}}^2$. Therefore,

$$\begin{aligned}\mathcal{L}(fw) &= w(\mathcal{L}_0^w)^*f - \operatorname{div}_{\mathcal{H}}(f\nabla_{\mathcal{H}}w) \\ &= -wf - \operatorname{div}_{\mathcal{H}}(f\nabla_{\mathcal{H}}w) \in S_{-1,\operatorname{loc}}^2.\end{aligned}\tag{2.2}$$

Hence, by Lemma 1.2, (iii), $fw \in S_{1,\operatorname{loc}}^2$. Thus

$$\nabla_{\mathcal{H}}f = \nabla_{\mathcal{H}}(fw w^{-1}) = w^{-1}\nabla_{\mathcal{H}}(fw) - f\frac{\nabla_{\mathcal{H}}w}{w} \in L_{\operatorname{loc}}^2$$

since both w^{-1} and $\nabla_{\mathcal{H}}w$ belong to $L_{\operatorname{loc}}^\infty$. Hence $f \in S_{1,\operatorname{loc}}^2$.

Next, assume that $\mathcal{L}w \in L_{\operatorname{loc}}^Q$. Since $f \in S_{1,\operatorname{loc}}^2$, by Lemma 1.2, (i) it is enough to prove that $\mathcal{L}f \in L_{\operatorname{loc}}^2$. First we prove that $\mathcal{L}(fw) \in L_{\operatorname{loc}}^2$. Recall that S_1^2 embeds continuously in $L^{2Q/(Q-2)}$ (cf. [6, Theorem 4.17]). Thus $f \in L_{\operatorname{loc}}^{2Q/(Q-2)}$. By applying Hölder's inequality to the conjugate exponents $Q/2$ and $Q/(Q-2)$, we obtain that

$$f\mathcal{L}w \in L_{\operatorname{loc}}^2.\tag{2.3}$$

Thus, since $\nabla_{\mathcal{H}}f \in L_{\operatorname{loc}}^2$ and $\nabla_{\mathcal{H}}f \cdot \nabla_{\mathcal{H}}w \in L_{\operatorname{loc}}^2$, we get that

$$\operatorname{div}_{\mathcal{H}}(f\nabla_{\mathcal{H}}w) = \nabla_{\mathcal{H}}f \cdot \nabla_{\mathcal{H}}w - f\mathcal{L}w \in L_{\operatorname{loc}}^2.\tag{2.4}$$

Thus, by (2.2)

$$\mathcal{L}(fw) = -(\operatorname{div}_{\mathcal{H}}(f\nabla_{\mathcal{H}}w) + wf) \in L_{\operatorname{loc}}^2.$$

Since $\nabla_{\mathcal{H}}(fw) \in L_{\operatorname{loc}}^2$, by Lemma 1.2, (i) we get that $fw \in S_{2,\operatorname{loc}}^2$. Therefore, we can write

$$\mathcal{L}f = \mathcal{L}(fw w^{-1}) = \mathcal{L}(fw)w^{-1} + fw\mathcal{L}(w^{-1}) + 2\nabla_{\mathcal{H}}(fw) \cdot \nabla_{\mathcal{H}}(w^{-1}).$$

The first and the third term are in L_{loc}^2 for what we showed above. As for the second term, since $\mathcal{L}w \in L_{\operatorname{loc}}^Q$,

$$\begin{aligned}fw\mathcal{L}(w^{-1}) &= fw\frac{\mathcal{L}w}{w^2} - 2fw\frac{|\nabla_{\mathcal{H}}w|^2}{w^3} \\ &= f\frac{\mathcal{L}w}{w} - 2f\frac{|\nabla_{\mathcal{H}}w|^2}{w^2}\end{aligned}$$

Thus also the third term is in L_{loc}^2 , since $f\mathcal{L}w \in L_{\operatorname{loc}}^2$ by (2.3) and $w, \nabla_{\mathcal{H}}w$ are in $L_{\operatorname{loc}}^\infty$. This concludes the proof. \square

Proof of Theorem 2.3. We begin by proving that

$$\operatorname{Dom}(\mathcal{L}^w) = \{f \in \operatorname{Dom}(\mathfrak{t}) : \mathcal{L}_1^w f \in L^2(w)\}, \quad \mathcal{L}^w f = \mathcal{L}_1^w f \quad \forall f \in \operatorname{Dom}(\mathcal{L}^w). \tag{2.5}$$

The inclusion \supseteq follows from Lemma 2.5. Assume then $f \in \operatorname{Dom}(\mathcal{L}^w)$, and take $\phi \in C_c^\infty$. Then $w^{-1}\phi \in \operatorname{Dom}(\mathfrak{t})$, so that

$$\begin{aligned}\langle \mathcal{L}^w f, \phi \rangle &= (\mathcal{L}^w f, w^{-1}\phi)_{L^2(w)} \\ &= \mathfrak{t}(f, w^{-1}\phi) \\ &= \langle \nabla_{\mathcal{H}}f, \nabla_{\mathcal{H}}\phi - w^{-1}\phi\nabla_{\mathcal{H}}w \rangle \\ &= \langle \mathcal{L}f, \phi \rangle - \left\langle \frac{\nabla_{\mathcal{H}}w}{w} \cdot \nabla_{\mathcal{H}}f, \phi \right\rangle,\end{aligned}$$

and hence $\mathcal{L}^w f = \mathcal{L}_1^w f$. Then (2.5) holds, and it is now easy to see that $C_c^\infty \subseteq \operatorname{Dom}(\mathcal{L}^w)$; thus, \mathcal{L}^w is a self-adjoint extension of \mathcal{L}_0^w .

Case (a). Assume that $\nabla_{\mathcal{H}}w \in L_{\text{loc}}^\infty$ and $\mathcal{L}w \in L_{\text{loc}}^Q$. By [15, Theorem X.1] it suffices to prove that if $f \in \text{Dom}((\mathcal{L}_0^w)^*)$ satisfies $(\mathcal{L}_0^w)^*f + f = 0$, then $f = 0$. By Lemma 2.6 $f \in S_{2,\text{loc}}^2$. Thus

$$\nabla_{\mathcal{H}}(fw) = w\nabla_{\mathcal{H}}f + f\nabla_{\mathcal{H}}w, \quad \text{div}_{\mathcal{H}}(w\nabla_{\mathcal{H}}f) = -w\mathcal{L}f + \nabla_{\mathcal{H}}w \cdot \nabla_{\mathcal{H}}f$$

are in L_{loc}^2 . Therefore, for every $\phi \in C_c^\infty$,

$$\begin{aligned} ((\mathcal{L}_0^w)^*f, \phi)_{L^2(w)} &= (f, \mathcal{L}_0^w\phi)_{L^2(w)} \\ &= \langle fw, \mathcal{L}\phi \rangle - \langle f\nabla_{\mathcal{H}}w, \nabla_{\mathcal{H}}\phi \rangle \\ &= \langle \nabla_{\mathcal{H}}(fw), \nabla_{\mathcal{H}}\phi \rangle - \langle f\nabla_{\mathcal{H}}w, \nabla_{\mathcal{H}}\phi \rangle \\ &= \langle w\nabla_{\mathcal{H}}f, \nabla_{\mathcal{H}}\phi \rangle \\ &= -\langle \text{div}_{\mathcal{H}}(w\nabla_{\mathcal{H}}f), \phi \rangle \\ &= \langle w\mathcal{L}f - \nabla_{\mathcal{H}}w \cdot \nabla_{\mathcal{H}}f, \phi \rangle \\ &= \langle \mathcal{L}_1^w f, w\phi \rangle \\ &= (\mathcal{L}_1^w f, \phi)_{L^2(w)}. \end{aligned}$$

Hence,

$$\mathcal{L}_1^w f = (\mathcal{L}_0^w)^*f \in L^2(w).$$

We are now able to argue as Strichartz [18]. Take an approximate unity (ψ_k) for G as in Remark 2.4. Hence $\psi_k^2 f \in S_1^2(w)$ and

$$\begin{aligned} 0 &\geq -(\psi_k^2 f, f)_{L^2(w)} = (\psi_k^2 f, (\mathcal{L}_0^w)^*f)_{L^2(w)} \\ &= (\psi_k^2 f, \mathcal{L}_1^w f)_{L^2(w)} \\ &= (\nabla_{\mathcal{H}}(\psi_k^2 f), \nabla_{\mathcal{H}}f)_{L^2(w)} \\ &= (2\psi_k(\nabla_{\mathcal{H}}\psi_k)f, \nabla_{\mathcal{H}}f)_{L^2(w)} + (\psi_k^2 \nabla_{\mathcal{H}}f, \nabla_{\mathcal{H}}f)_{L^2(w)}. \end{aligned}$$

where the third equality holds by Lemma 2.5 (i). Therefore,

$$\begin{aligned} \|\psi_k \nabla_{\mathcal{H}}f\|_{L^2(w)}^2 &= \langle \psi_k^2 \nabla_{\mathcal{H}}f, \nabla_{\mathcal{H}}f \rangle_{L^2(w)} \\ &\leq -\langle 2\psi_k(\nabla_{\mathcal{H}}\psi_k)f, \nabla_{\mathcal{H}}f \rangle_{L^2(w)} \leq 2\|\psi_k \nabla_{\mathcal{H}}f\|_{L^2(w)}\|f \nabla_{\mathcal{H}}\psi_k\|_{L^2(w)}, \end{aligned}$$

so that

$$\|\psi_k \nabla_{\mathcal{H}}f\|_{L^2(w)} \leq 2\|\nabla_{\mathcal{H}}\psi_k\|_{L^\infty(w)}\|f\|_{L^2(w)}.$$

By Fatou's lemma, finally,

$$\|\nabla_{\mathcal{H}}f\|_{L^2(w)} \leq \liminf_{k \rightarrow \infty} \|\psi_k \nabla_{\mathcal{H}}f\|_{L^2(w)} \leq \lim_{k \rightarrow \infty} \frac{2}{k} \|\nabla_{\mathcal{H}}\psi_k\|_{L^\infty(w)}\|f\|_{L^2(w)} = 0,$$

whence $\nabla_{\mathcal{H}}f = 0$. Therefore, $f = 0$ and Case (a) is proved.

Case (b). Assume that $\frac{\nabla_{\mathcal{H}}w}{(1+N)w} \in L^\infty$. To prove that \mathcal{L}_0^w is essentially self-adjoint it suffices to show that C_c^∞ is dense in $\text{Dom}(\mathcal{L}^w)$. First we show that the space of functions in $\text{Dom}(\mathcal{L}^w)$ with compact support is dense in $\text{Dom}(\mathcal{L}^w)$.

Let $f \in \text{Dom}(\mathcal{L}^w)$ and take an approximate unity (ψ_k) as in Remark (2.4). Then $\psi_k f \in \text{Dom}(\mathcal{L}^w)$ for all k by (2.5), since

$$\mathcal{L}_1^w(\psi_k f) = (\mathcal{L}_0^w \psi_k)f + \psi_k(\mathcal{L}_1^w f) - 2\nabla_{\mathcal{H}}\psi_k \cdot \nabla_{\mathcal{H}}f, \quad (2.6)$$

is in $L^2(w)$. Moreover $\mathcal{L}_1^w(\psi_k f)$ tends to $\mathcal{L}_1^w f$ in L^2 . Indeed, $\lim_k \psi_k(\mathcal{L}_1^w f) = \mathcal{L}_1^w f$ in $L^2(w)$ by dominated convergence and $\lim_k \|\nabla_{\mathcal{H}}\psi_k \cdot \nabla_{\mathcal{H}}f\|_{L^2(w)} = 0$ since $\|\nabla_{\mathcal{H}}\psi_k\|_\infty \leq C/k$. Thus it remains only to prove that

$$(\mathcal{L}_0^w \psi_k)f = (\mathcal{L}\psi_k)f - w^{-1}\nabla_{\mathcal{H}}w \cdot \nabla_{\mathcal{H}}\psi_k f$$

tends to zero in $L^2(w)$. Indeed, on the one hand, $\|\mathcal{L}\psi_k\|_\infty \leq Ck^{-2}$. On the other hand, since the support of $\nabla_{\mathcal{H}}\psi_k$ is contained in $A_k = \{x : k \leq N(x) \leq 2k\}$, we have that $k \sim (1+N)$ on the support of $\nabla_{\mathcal{H}}\psi_k$. Thus $\|w^{-1}\nabla_{\mathcal{H}}w \cdot \nabla_{\mathcal{H}}\psi_k f\|_{L^2(w)} \leq C\|f\|_{L^2(A_k, w)}$, which tends to zero.

It remains only to prove that if $f \in \text{Dom}(\mathcal{L}^w)$ has compact support then there exists a sequence (f_k) of functions in C_c^∞ such that $\lim_k f_k = f$ and $\lim_k \mathcal{L}_0^w f_k = \mathcal{L}^w f$ in $L^2(w)$. To this end, define $f_k = \eta_k * f$, where (η_k) is an approximate identity for the convolution in C_c^∞ and observe that

$$\begin{aligned} \mathcal{L}_0^w(\eta_k * f) &= \mathcal{L}(\eta_k * f) - \frac{\nabla_{\mathcal{H}}w}{w} \cdot \nabla_{\mathcal{H}}(\eta_k * f) \\ &= \eta_k * \mathcal{L}f - \frac{\nabla_{\mathcal{H}}w}{w} \cdot \eta_k * (\nabla_{\mathcal{H}}f). \end{aligned}$$

The conclusion follows, since for functions supported in a fixed compact set convergence in L^2 is equivalent to convergence in $L^2(w)$. This completes the proof. \square

The following corollary specializes the result of Theorem 2.3 to weights of the form $w_\alpha = e^{-N^\alpha}$, where N is any homogeneous norm on G and $\alpha > 0$.

Corollary 2.7. \mathcal{L}^{w_α} is a self-adjoint extension of $\mathcal{L}_0^{w_\alpha}$ for every $\alpha > 0$. When $\alpha \geq 1$, \mathcal{L}^{w_α} is the unique self-adjoint extension of $\mathcal{L}_0^{w_\alpha}$.

Proof. Outside the origin

$$\nabla_{\mathcal{H}}w_\alpha = -\alpha w_\alpha N^{\alpha-1} \nabla_{\mathcal{H}}N, \quad (2.7)$$

and

$$\mathcal{L}w_\alpha = w_\alpha \left[-\alpha^2 N^{2\alpha-2} |\nabla_{\mathcal{H}}N|^2 + \alpha(\alpha-1) N^{\alpha-2} |\nabla_{\mathcal{H}}N|^2 + \alpha N^{\alpha-1} \mathcal{L}N \right]. \quad (2.8)$$

Since the right hand sides of both identities are in L_{loc}^1 , it is not hard to see that they coincide with $\nabla_{\mathcal{H}}w_\alpha$ and $\mathcal{L}w_\alpha$, respectively, in the sense of distributions on G . Thus the weight w_α is admissible for all $\alpha > 0$. Moreover,

- (a) if $\alpha > 1$ then $\nabla_{\mathcal{H}}w_\alpha \in L_{\text{loc}}^\infty$ and $\mathcal{L}w_\alpha \in L_{\text{loc}}^Q$;
- (b) $\nabla_{\mathcal{H}}w_1 \in L_{\text{loc}}^\infty$ and $w_1^{-1} \nabla_{\mathcal{H}}w_1 \in L^\infty$.

The conclusion follows by Theorem 2.3. \square

3. SCHRÖDINGER OPERATORS

Let w be an admissible weight such that $\mathcal{L}w \in L_{\text{loc}}^1$. Instead of dealing directly with the weighted operator \mathcal{L}^w , it is often more convenient to consider a Schrödinger operator unitarily equivalent to it. Indeed, if $U: L^2(w) \rightarrow L^2$ is the unitary map defined by $Uf := f\sqrt{w}$, then \mathcal{L}^w is unitarily equivalent to $U\mathcal{L}^wU^{-1}$ with domain $U\text{Dom}(\mathcal{L}^w)$. If the weight w is smooth then U maps C_c^∞ onto itself and a straightforward computation gives

$$U\mathcal{L}^wU^{-1}\phi = (\mathcal{L} + V)\phi \quad \forall \phi \in C_c^\infty, \quad (3.1)$$

where

$$V = -\frac{1}{4} \frac{|\nabla_{\mathcal{H}}w|^2}{w^2} - \frac{1}{2} \frac{\mathcal{L}w}{w}. \quad (3.2)$$

If the operator $\mathcal{L} + V$ is essentially self-adjoint on C_c^∞ then the identity (3.1) completely determines \mathcal{L}^w and the study of the spectrum of \mathcal{L}^w can be reduced to that of the closure of $\mathcal{L} + V$. If the weight w is not smooth matters become more delicate because, on the one hand, the derivatives in (3.2) must be interpreted in the sense of distributions and, on the other hand, one must be careful in identifying the domains of the operators.

The following lemma shows that the identity (3.1) still holds in the distributional sense if we assume sufficient regularity of the weight w .

Lemma 3.1. *If $\nabla_{\mathcal{H}}w \in L_{\text{loc}}^\infty$ and $\mathcal{L}w \in L_{\text{loc}}^2$, then $V \in L_{\text{loc}}^2$ and*

$$U^{-1}\phi \in \text{Dom}(\mathcal{L}^w), \quad U\mathcal{L}^wU^{-1}\phi = \mathcal{L}\phi + V\phi \quad \forall \phi \in C_c^\infty.$$

Proof. Let ϕ be a function in C_c^∞ . A straightforward computation shows that $U^{-1}\phi \in S_1^2(w)$ and $\mathcal{L}_1^wU^{-1}\phi \in L^2(w)$. Thus $U^{-1}\phi \in \text{Dom}(\mathcal{L}^w)$, by Theorem 2.3. The identity $U\mathcal{L}^wU^{-1}\phi = \mathcal{L}\phi + V\phi$ follows easily. \square

The following theorem is an analogue of a classical result for the Laplacian on \mathbb{R}^n (see Kato [10]). It will be used in Proposition 3.4 to prove that, when the weight w is sufficiently regular, the operator \mathcal{L}^w is unitarily equivalent to the closure of $\mathcal{L} + V$ on C_c^∞ .

Theorem 3.2. *Let $V \in L_{\text{loc}}^2$, and suppose that V is essentially bounded from below. Then $\mathcal{L} + V$, with domain C_c^∞ , is essentially self-adjoint on L^2 .*

Proof. Up to adding a constant, we can assume that $V \geq 0$. By [15, Theorem X.1] it is enough to prove that, if $(\mathcal{L} + V)^*f + f = 0$ for some $f \in \text{Dom}((\mathcal{L} + V)^*)$, then $f = 0$. This holds in particular if we show that, if $(\mathcal{L} + V)f + f = 0$ in the sense of distributions for some $f \in L^2$, then $f = 0$.

Suppose, then,

$$\mathcal{L}f + Vf + f = 0.$$

Since $f \in L^2$ and $V \in L_{\text{loc}}^2$, $Vf \in L_{\text{loc}}^1$. Moreover, $f \in L_{\text{loc}}^1$ and hence $\mathcal{L}f = -(\mathcal{L}f + Vf) \in L_{\text{loc}}^1$. Therefore, by Proposition 1.3

$$\mathcal{L}|f| \leq \text{Re} \left[\overline{\text{sign}(f)} \mathcal{L}f \right] = -\text{Re} \left[\overline{\text{sign}(f)} (V + 1)f \right] = -(V + 1)|f|,$$

so that

$$(\mathcal{L} + I)|f| \leq -V|f| \leq 0.$$

In particular, for every nonnegative $\phi \in C_c^\infty$,

$$\langle |f|, (\mathcal{L} + I)\phi \rangle = \langle (\mathcal{L} + I)|f|, \phi \rangle \leq 0. \quad (3.3)$$

More generally, if g is a nonnegative function in $\text{Dom}(\mathcal{L})$, by approximating g with a sequence (ϕ_k) of nonnegative functions in C_c^∞ which tends to g in $\text{Dom}(\mathcal{L})^1$, we obtain that

$$\langle |f|, (\mathcal{L} + I)g \rangle = \lim_{k \rightarrow \infty} \langle |f|, (\mathcal{L} + I)\phi_k \rangle \leq 0.$$

In particular, if

$$g = (\mathcal{L} + I)^{-1}\psi = \int_0^\infty e^{-s}e^{-s\mathcal{L}}\psi ds,$$

where ψ is a nonnegative function in C_c^∞ , then $g \in \text{Dom}(\mathcal{L})$ and is nonnegative, since the semigroup $e^{-s\mathcal{L}}$ is positivity preserving (cf. [13, Theorem 2.7]). Thus

$$\langle |f|, \psi \rangle \leq 0 \quad \forall \psi \in C_c^\infty, \psi \geq 0.$$

It follows that $f = 0$, and this concludes the proof. \square

Notation 3.3. When T is an operator with domain $\text{Dom}(T)$ and $\mathcal{D} \subseteq \text{Dom}(T)$, we denote by (T, \mathcal{D}) the restriction of T to \mathcal{D} .

¹It suffices to choose $\phi_k = \psi_k(\eta_k * g)$, where (ψ_k) is a nonnegative approximate unity and (η_k) is a nonnegative approximate identity.

Proposition 3.4. *Let w be an admissible weight such that $\nabla_{\mathcal{H}} w \in L_{\text{loc}}^\infty$ and $\mathcal{L}w \in L_{\text{loc}}^2$. Suppose that V in (3.2) is bounded from below. Then \mathcal{L}^w is unitarily equivalent to the closure of $(\mathcal{L} + V, C_c^\infty)$.*

Proof. If ϕ is a function in C_c^∞ then $U^{-1}\phi \in \text{Dom}(\mathcal{L}^w)$, by Lemma 3.1. Hence $C_c^\infty \subset U\text{Dom}(\mathcal{L}^w)$ and $(U\mathcal{L}^wU^{-1}, U\text{Dom}(\mathcal{L}^w))$ is a self-adjoint extension of $(\mathcal{L} + V, C_c^\infty)$. Since the latter operator is essentially self-adjoint by Theorem 3.2, $(U\mathcal{L}^wU^{-1}, U\text{Dom}(\mathcal{L}^w))$ is the closure of $(\mathcal{L} + V, C_c^\infty)$. This proves the proposition. \square

4. DISCRETENESS OF THE SPECTRUM ON MÉTIVIER GROUPS

This section is based on a result of Simon [17] on the discreteness of the spectrum of the Schrödinger operator $-\Delta + V$ on \mathbb{R}^n , when the sublevel sets of the potential V are *polynomially thin*. Simon's definition of polynomial thinness can be adapted to the setting of stratified groups as follows. For a subset E of G , we write $|E|$ to denote its measure with respect to the measure dy .

Definition 4.1. Given $\ell > 0$, a set $\Omega \subseteq G$ is said to be ℓ -polynomially thin if

$$\int_{\Omega} |\Omega \cap B(y, r)|^\ell dy < \infty$$

for every $r > 0$. Here $B(y, r)$ is the ball induced by the left-invariant metric associated with any homogeneous norm on G .

Theorem 4.2. *Let V be a potential bounded from below. Assume that for every $M > 0$ there is $\ell > 0$ such that $\Omega_M := \{y \in G : V(y) \leq M\}$ is ℓ -polynomially thin. Then there exists a self-adjoint extension of $(\mathcal{L} + V, C_c^\infty)$ with purely discrete spectrum.*

The proof is a simple adaptation to stratified groups of [17, Theorem 3]. Indeed, Simon's proof relies only on some properties of the heat kernel on \mathbb{R}^n which are true also for the heat kernel on G . This theorem holds on general stratified groups, but henceforth we shall restrict to the particular case where G is a Métivier group.

Definition 4.3. (cf. [11]) Let \mathfrak{g} be a two-step Lie algebra with centre \mathfrak{z} , and denote by \mathfrak{h} a complement of \mathfrak{z} in \mathfrak{g} . Let G be the connected, simply connected Lie group associated with \mathfrak{g} . We say that G is an H-type group in the sense of Métivier, or simply a Métivier group, if for every $\eta \in \mathfrak{z}^*$ the skew-symmetric bilinear form on \mathfrak{h}

$$B_\eta(X, Y) := \eta([X, Y])$$

is non-degenerate whenever $\eta \neq 0$.

From now on, G will be a Métivier group. We shall endow \mathfrak{g} with some inner product (\cdot, \cdot) and choose $\mathfrak{h} := \mathfrak{z}^\perp$, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ is an orthogonal stratification (1.1) of \mathfrak{g} (cf. [2, Section 3.7]).

It is very convenient to realize G as $\mathbb{R}^{2n} \times \mathbb{R}^m$, for some $n, m \in \mathbb{N}$, via the exponential map. More precisely, we shall denote by (x, t) the elements of G , where $x \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}^m$. We denote by (e_1, \dots, e_{2n}) and (u_1, \dots, u_m) the standard basis of \mathbb{R}^{2n} and \mathbb{R}^m respectively. Under this identification, the Haar measure dy is the Lebesgue measure. If we define, for $T \in \mathfrak{z}$, a map $J_T : \mathfrak{h} \rightarrow \mathfrak{h}$ such that

$$(J_TX, Y) = (T, [X, Y]) \quad \forall X, Y \in \mathfrak{h},$$

then the exponential map identifies the maps $\{J_T : T \in \mathfrak{z}\}$ with $2n \times 2n$ skew symmetric matrices $\{J_t : t \in \mathbb{R}^m\}$ and the group law on G is

$$(x, t) \cdot (x', t') = \left(x + x', t + t' + \frac{1}{2} \sum_{k=1}^m (J_{u_k} x, x') u_k \right)$$

for every $(x, t), (x', t') \in \mathbb{R}^{2n} \times \mathbb{R}^m$. By definition of Métivier group, the map J_t is non-degenerate whenever $t \neq 0$.

A basis of left-invariant vector fields for \mathfrak{g} is

$$X_j = \partial_{x_j} + \frac{1}{2} \sum_{k=1}^m (J_{u_k} x, e_j) \partial_{t_k}, \quad j = 1, \dots, 2n; \quad T_k = \partial_{t_k}, \quad k = 1, \dots, m.$$

In particular, $(X_j)_{1 \leq j \leq 2n}$ is a basis for the first layer $\mathfrak{h} \cong \mathbb{R}^{2n}$.

From now on, we fix the homogeneous norm

$$N(x, t) = \left(|x|^4 + 16 |t|^2 \right)^{1/4}$$

and the associated left invariant (quasi) distance d . A pseudo-triangle inequality for N holds,

$$N((x, t) \cdot (\xi, \tau)) \leq \gamma [N(x, t) + N(\xi, \tau)]$$

for some $\gamma > 0$ which depends on N . In the particular case when G is an H-type group with respect to the inner product (\cdot, \cdot) , such norm N is the norm appearing in the fundamental solution of the sub-Laplacian [8].

Our main theorem is the following, and its proof will occupy the remainder of the paper.

Theorem 4.4. *If $0 < \alpha < 1$ no self-adjoint extension of $\mathcal{L}_0^{w_\alpha}$ has purely discrete spectrum. If $1 \leq \alpha \leq 2$, \mathcal{L}^{w_α} does not have purely discrete spectrum. If $\alpha > 2$, the spectrum of \mathcal{L}^{w_α} is purely discrete.*

To prove Theorem 4.4 we reduce matters to studying the spectrum of the Schrödinger operator $\mathcal{L} + V_\alpha$ obtained by conjugating \mathcal{L}^{w_α} with the isometry $U_\alpha: L^2(w_\alpha) \ni f \mapsto f\sqrt{w_\alpha} \in L^2$, as explained in the previous section.

Lemma 4.5. *Let $\mathcal{L} + V_\alpha$ be the Schrödinger operator associated with \mathcal{L}^{w_α} . Then, for every $(x, t) \in G^*$,*

$$N^{2\alpha-4}|x|^2 \left(c_{\alpha,1} - \frac{c_{\alpha,2}}{N(x, t)^\alpha} \right) \leq V_\alpha(x, t) \leq N^{2\alpha-4}|x|^2 \left(c_{\alpha,3} - \frac{c_{\alpha,4}}{N(x, t)^\alpha} \right) \quad (4.1)$$

for some $c_{\alpha,1}, c_{\alpha,2}, c_{\alpha,3} > 0$ and $c_{\alpha,4} \in \mathbb{R}$. If $\alpha \geq 2$, then V_α is bounded from below.

Proof. Observe first that

$$\begin{aligned} V_\alpha &= -\frac{1}{4} \frac{|\nabla_{\mathcal{H}} w_\alpha|^2}{w_\alpha^2} - \frac{1}{2} \frac{\mathcal{L} w_\alpha}{w_\alpha} \\ &= \frac{1}{4} \alpha^2 N^{2\alpha-2} |\nabla_{\mathcal{H}} N|^2 - \frac{1}{2} \alpha(\alpha-1) N^{\alpha-2} |\nabla_{\mathcal{H}} N|^2 + \frac{1}{2} \alpha N^{\alpha-1} \mathcal{L} N \end{aligned}$$

on G^* . It is therefore enough to compute $|\nabla_{\mathcal{H}} N|^2$ and $\mathcal{L} N$. Easy computations show that

$$\begin{aligned} |\nabla_{\mathcal{H}} N|^2(x, t) &= \frac{|x|^2}{N^6} \left(|x|^4 + 16|t|^2 |J_{\text{sign}(t)} \text{sign}(x)|^2 \right), \\ \mathcal{L} N(x, t) &= \frac{3}{N} |\nabla_{\mathcal{H}} N|^2 - \frac{|x|^2}{N^3} \left(2 + 2n + 2 \sum_{k=1}^m |J_{u_k} \text{sign}(x)|^2 \right). \end{aligned}$$

Let c_0 and C_0 be the minimum and maximum, respectively, of $|J_t x|^2$ as $|x| = |t| = 1$. Since J_t is non-degenerate for $t \neq 0$,

$$\min\{|J_t x| : x \in \mathbb{R}^{2n}, t \in \mathbb{R}^m, |x| = |t| = 1\} > 0,$$

hence $c_0 > 0$. Let $C := \max(C_0, 1)$ and $c := \min(c_0, 1)$. Thus

$$c \frac{|x|^2}{N^2} \leq |\nabla_{\mathcal{H}} N|^2 \leq C \frac{|x|^2}{N^2}$$

and

$$\frac{|x|^2}{N^3} (3c - (2n + 2) - 2mC_0) \leq \mathcal{L}N \leq \frac{|x|^2}{N^3} (3C - (2n + 2) - 2mc_0).$$

Therefore (4.1) holds with

$$\begin{aligned} c_{\alpha,1} &= \frac{c\alpha^2}{4}, & c_{\alpha,2} &= \frac{C\alpha^2}{2} - \frac{\alpha}{2}(4c - 2n - 2 - 2mC_0), \\ c_{\alpha,3} &= \frac{C\alpha^2}{4}, & c_{\alpha,4} &= \frac{c\alpha^2}{2} - \frac{\alpha}{2}(4C - 2n - 2 - 2mc_0). \end{aligned}$$

Since $4c - 2n - 2 \leq 0$, it is easily seen that $c_{\alpha,2} > 0$. The boundedness from below of V_α when $\alpha \geq 2$ is a straightforward consequence of (4.1). \square

Remark 4.6. When J_t is $|t|$ -times an isometry for every $t \in \mathbb{R}^m$ and hence G is an H-type group, by (4.1) we get

$$V_\alpha(x, t) = \frac{1}{4}\alpha^2 N(x, t)^{2\alpha-4} |x|^2 - \frac{1}{2}\alpha(Q + \alpha - 2)N(x, t)^{\alpha-4} |x|^2$$

as already shown by Inglis [7].

Corollary 4.7. *For every $\alpha \geq 2$, $(\mathcal{L} + V_\alpha, C_c^\infty)$ is essentially self-adjoint and \mathcal{L}^{w_α} is unitarily equivalent to the closure of $(\mathcal{L} + V_\alpha, C_c^\infty)$.*

Proof. If $\alpha \geq 2$, then $\nabla_{\mathcal{H}} w_\alpha \in L_{\text{loc}}^\infty$ and $\mathcal{L}w_\alpha \in L_{\text{loc}}^Q \subset L_{\text{loc}}^2$ by (2.7) and (2.8), while V_α is bounded from below by Lemma 4.5. The conclusion follows from Proposition 3.4 \square

Remark 4.8. When $0 < \alpha < 2$, V_α is not bounded from below, and we do not know whether $(\mathcal{L} + V_\alpha, C_c^\infty)$ is essentially self-adjoint. Therefore we cannot conclude that \mathcal{L}^{w_α} is unitarily equivalent to the closure of $(\mathcal{L} + V_\alpha, C_c^\infty)$. However, since the weight w_α is smooth in G^* , the operators $(\mathcal{L}^{w_\alpha}, C_c^\infty(G^*))$ and $(\mathcal{L} + V_\alpha, C_c^\infty(G^*))$ are unitarily equivalent. As we shall see this will suffice to show that no self-adjoint extension of $(\mathcal{L}^{w_\alpha}, C_c^\infty(G^*))$ has discrete spectrum when $0 < \alpha \leq 2$.

Definition 4.9. Given $\alpha > 0$ and $M > 0$, we set

$$\Omega_{\alpha, M} := \{(x, t) \in G : V_\alpha(x, t) \leq M\}.$$

Lemma 4.10. *Let $\alpha > 2$. Then, for every $M > 0$ and $r > 0$ there exists $k = k(\alpha, M, r)$ such that, for every $(x, t) \in \Omega_{\alpha, M}$ with $|t| > k$,*

$$|\Omega_{\alpha, M} \cap B((x, t), r)| \leq C(\alpha, M, r) |t|^{n(2-\alpha)}.$$

Proof. Since $V_\alpha \rightarrow +\infty$ when $|x| \rightarrow +\infty$, for every M there exists $c = c(\alpha, M) > 0$ such that

$$|x| \leq c(\alpha, M) \quad \forall (x, t) \in \Omega_{\alpha, M}. \quad (4.2)$$

Let $R = \gamma(r + c(\alpha, M))$. Then for every $(x, t) \in \Omega_{\alpha, M}$ the ball $B((x, t), r)$ is contained in $B((0, t), R)$ because the quasi-distance between the centres is $|x| \leq c(\alpha, M)$. Thus

$$\Omega_{\alpha, M} \cap B((x, t), r) \subseteq \Omega_{\alpha, M} \cap B((0, t), R) \quad \forall (x, t) \in \Omega_{\alpha, M}.$$

Since $B((0, t), R)$ is contained in the cylinder $C_R(0, t) = \{(\xi, \tau) : |\xi| \leq R, |\tau - t| \leq R^2\}$, it suffices to estimate the measure of $\Omega_{\alpha, M} \cap C_R(0, t)$. If $k > R^2$, then for all $|t| > k$ and $(\xi, \tau) \in C_R(0, t)$

$$N(\xi, \tau) \geq |\tau|^{1/2} \geq (|t| - R^2)^{1/2} > (k - R^2)^{1/2}.$$

Thus, if we choose k sufficiently large, the quantity $\left(c_{\alpha,1} - \frac{c_{\alpha,2}}{N(\xi,\tau)^\alpha}\right)$ is bounded from below by a positive constant $K = K(\alpha, R, k)$. Hence

$$\begin{aligned} |\Omega_{\alpha,M} \cap C_R(0,t)| &\leq \left| \left\{ (\xi, \tau) : |\tau|^{\alpha-2} |\xi|^2 \leq \frac{M}{K}, |\xi| \leq R, |t - \tau| \leq R^2 \right\} \right| \\ &\leq \int_{|\tau-t| \leq R^2} \int_{|\xi|^2 \leq \frac{M}{K} |\tau|^{2-\alpha}} d\xi d\tau \\ &\lesssim \int_{B(t, R^2)} \frac{1}{|\tau|^{n(\alpha-2)}} d\tau \lesssim |t|^{n(2-\alpha)}. \end{aligned}$$

This completes the proof. \square

Proposition 4.11. *If $0 < \alpha \leq 2$, then no self-adjoint extension of $(\mathcal{L} + V_\alpha, C_c^\infty(G^*))$ has purely discrete spectrum. If $\alpha > 2$, then the unique self-adjoint extension of $(\mathcal{L} + V_\alpha, C_c^\infty)$ has purely discrete spectrum.*

Proof. Suppose that $0 < \alpha \leq 2$, and take any self-adjoint extension T_α of $(\mathcal{L} + V_\alpha, C_c^\infty(G^*))$. Fix any λ in the resolvent set of T_α . We shall prove that $(\lambda I + T_\alpha)^{-1}$ is non-compact. Consider a nonzero C^∞ function $\psi : G \rightarrow \mathbb{C}$ supported in $B(e, 1)$ and for each $n \in \mathbb{N}$ let ψ_n be the left translate of ψ by $(0, n)$, i.e.

$$\psi_n(x, t) := L_{(0,n)}\psi(x, t) = \psi((0, -n)(x, t)) = \psi(x, t - n),$$

Then $\text{supp}(\psi_n) \subseteq B((0, n), 1)$; hence $\psi_n \in C_c^\infty(G^*) \subseteq \text{Dom}(T_\alpha)$ for every $n \geq 1$. Denote by \mathcal{C} the set $\{(x, t) : |x| \leq 1, N(x, t) \geq 1\}$; then $C := \|V_\alpha \chi_{\mathcal{C}}\|_\infty$ is finite by Lemma 4.5, and $\text{supp}(\psi_n) \subseteq \mathcal{C}$ for $n \geq 2$. Define $f_n := (\lambda I + T_\alpha)\psi_n = (\lambda I + \mathcal{L} + V_\alpha)\psi_n$ for $n \geq 2$. Then,

$$\begin{aligned} \|f_n\|_2 &= \|(\lambda I + \mathcal{L} + V_\alpha)\psi_n\|_2 \leq |\lambda| \|\psi_n\|_2 + \|\mathcal{L}\psi_n\|_2 + \|V_\alpha \chi_{\mathcal{C}}\|_\infty \|\psi_n\|_2 \\ &\leq (|\lambda| + C) \|\psi\|_2 + \|\mathcal{L}\psi\|_2, \end{aligned}$$

since

$$\|\mathcal{L}\psi_n\|_2 = \|\mathcal{L}L_{(0,n)}\psi\|_2 = \|L_{(0,n)}\mathcal{L}\psi\|_2 = \|\mathcal{L}\psi\|_2,$$

by the left-invariance of \mathcal{L} . Thus (f_n) is a bounded sequence in L^2 such that $(\lambda I + T_\alpha)^{-1}f_n = \psi_n$ does not admit any convergent subsequence, since

$$\|\psi_n - \psi_m\|_2^2 = \|\psi_n\|_2^2 + \|\psi_m\|_2^2 = 2\|\psi\|_2^2.$$

Hence the operator $(\lambda I + T_\alpha)^{-1}$ is non-compact, so that T_α cannot have purely discrete spectrum (cf. [12, Theorem 11.3.13]).

Let now $\alpha > 2$. Since V_α is bounded from below, by Theorem 4.2 it is enough to prove that for every M the sublevel set $\Omega_{\alpha,M}$ is ℓ -polynomially thin for a suitable $\ell > 0$. Choose then $r > 0$. First, for any $\ell > 0$, write

$$\int_{\Omega_{\alpha,M}} |\Omega_{\alpha,M} \cap B((x, t), r)|^\ell dt dx = \left(\int_{\Omega_{\alpha,M}^1} + \int_{\Omega_{\alpha,M}^2} \right) |\Omega_{\alpha,M} \cap B((x, t), r)|^\ell dt dx$$

where

$$\Omega_{\alpha,M}^1 := \Omega_{\alpha,M} \cap \{|t| \leq k\}, \quad \Omega_{\alpha,M}^2 := \Omega_{\alpha,M} \cap \{|t| > k\},$$

and $k = k(\alpha, M, r)$ is that of Lemma 4.10.

The integral over $\Omega_{\alpha,M}^1$ is finite for every ℓ since the integrand is bounded and $\Omega_{\alpha,M}^1$ has finite measure by (4.2). By Lemma 4.10 and (4.2) we get

$$\int_{\Omega_{\alpha,M}^2} |\Omega_{\alpha,M} \cap B((x, t), r)|^\ell \leq C(\alpha, M, r) \int_{\{|x| \leq c\}} \int_{\{|t| > k\}} t^{\ell n(2-\alpha)} dt dx$$

which is finite for $\ell > \frac{m}{n(\alpha-2)}$. This completes the proof. \square

We can now prove Theorem 4.4.

Proof of Theorem 4.4. Let $\alpha \in (0, 2]$ and let $(T_\alpha, \mathcal{D}_\alpha)$ be a self-adjoint extension of $\mathcal{L}_0^{w_\alpha}$. Then $(T_\alpha, \mathcal{D}_\alpha)$ is also a self-adjoint extension of $(\mathcal{L}_0^{w_\alpha}, C_c^\infty(G^*))$ and therefore $(U_\alpha T_\alpha U_\alpha^{-1}, U_\alpha \mathcal{D}_\alpha)$ is a self-adjoint extension of $(\mathcal{L} + V_\alpha, C_c^\infty(G^*))$, since $U_\alpha \mathcal{D}_\alpha \supseteq C_c^\infty(G^*)$. Since $U_\alpha T_\alpha U_\alpha^{-1}$ does not have purely discrete spectrum by Proposition 4.11, neither does T_α .

Let $\alpha \in (2, \infty)$. By Lemma 4.5, V_α is continuous and bounded from below. Therefore, \mathcal{L}^{w_α} is unitarily equivalent to the closure of the operator $(\mathcal{L} + V_\alpha, C_c^\infty)$ by Proposition 3.4. Since by Proposition 4.11 the closure of $(\mathcal{L} + V_\alpha, C_c^\infty)$ has purely discrete spectrum, the same holds for \mathcal{L}^{w_α} . \square

Corollary 4.12. *Let $\alpha > 2$. Then, the semigroup $e^{-t\mathcal{L}^{w_\alpha}}$ is compact on $L^p(w_\alpha)$ for all $t > 0$ and $1 < p < \infty$, and the spectrum of \mathcal{L}^{w_α} on $L^p(w_\alpha)$ is independent of p for $1 < p < \infty$.*

Proof. The semigroup $(e^{-t\mathcal{L}^{w_\alpha}})_{t>0}$ is Markovian, since it is associated with a Dirichlet form. Thus it is contractive on L^∞ and positivity-preserving. Since the semigroup is symmetric and compact on $L^2(w)$ by Theorem 4.4, this proves the statement by [4, Theorem 1.6.3]. \square

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